

## THE WILLMORE FLOW WITH SMALL INITIAL ENERGY

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### Abstract

We consider the  $L^2$  gradient flow for the Willmore functional. In [5] it was proved that the curvature concentrates if a singularity develops. Here we show that a suitable blowup converges to a nonumbilic (compact or noncompact) Willmore surface. Furthermore, an  $L^\infty$  estimate is derived for the tracefree part of the curvature of a Willmore surface, assuming that its  $L^2$  norm (the Willmore energy) is locally small. One consequence is that a properly immersed Willmore surface with restricted growth of the curvature at infinity and small total energy must be a plane or a sphere. Combining the results we obtain long time existence and convergence to a round sphere if the total energy is initially small.

### 1. Introduction

For a closed, immersed surface  $f : \Sigma \rightarrow \mathbb{R}^n$  the Willmore functional (as introduced initially by Thomsen [11]) is

$$(1) \quad \mathcal{W}(f) = \int_{\Sigma} |A^\circ|^2 d\mu,$$

where  $A^\circ = A - \frac{1}{2}g \otimes H$  denotes the tracefree part of the second fundamental form  $A = D^2 f^\perp$  and  $\mu$  is the induced area measure. The associated Euler-Lagrange operator is

$$(2) \quad \mathbf{W}(f) = \Delta H + Q(A^\circ)H.$$

Here  $H$  is the mean curvature vector and  $Q(A^\circ)$  acts linearly on normal vectors along  $f$  by the formula (using summation with respect to a  $g$ -orthonormal basis  $\{e_1, e_2\}$ )

$$(3) \quad Q(A^\circ)\phi = A^\circ(e_i, e_j)\langle A^\circ(e_i, e_j), \phi \rangle.$$

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In (2) the Laplace operator  $\Delta\phi = -\nabla^*\nabla\phi$  is understood with respect to the connection  $\nabla_X\phi = (D_X\phi)^\perp$  on normal vector fields along  $f$ , where  $\nabla^*$  denotes the formal adjoint of  $\nabla$ .

In this paper we continue our study from [5] of the  $L^2$  gradient flow for (1), briefly called the Willmore flow, which is the fourth order, quasilinear geometric evolution equation

$$(4) \quad \partial_t f = -\mathbf{W}(f).$$

As a main result we have shown in [5] that the existence time is bounded from below in terms of the concentration of the measure  $f_\#(\mu_\perp|A|^2)$  in  $\mathbb{R}^n$  at time  $t = 0$ . Here we study the operator (2) and the flow (4) under the assumption that  $\mathcal{W}(f)$  is — either locally or globally — small. This condition is natural from the variational point of view and may be interpreted geometrically by saying that the deviation of  $f$  from being round is small in an averaged sense. One of our results is:

**Theorem 5.1.** *There exists  $\varepsilon_0(n) > 0$  such that if at time  $t = 0$  we have  $\mathcal{W}(f_0) < \varepsilon_0$ , then the Willmore flow exists smoothly for all times and converges to a round sphere.*

The smallness assumption implies, if  $\varepsilon_0$  is not too big, that  $\Sigma$  is topologically a sphere and that  $f$  is an embedding (see [13] for the case  $n = 3$ ). Moreover, any sequence  $f_k$  with  $\mathcal{W}(f_k) \rightarrow 0$  subconverges, after appropriate translation and rescaling, to some round sphere in the sense of both Hausdorff distance and measure [8]. However the  $f_k$  need not be graphs over the limit sphere, as can be seen by modifying Example 1 in [12]. At present we do not know an example ruling out the possibility of dropping the smallness condition in Theorem 5.1 entirely; in any case it is desirable to replace the number  $\varepsilon_0$  by a more explicit constant.<sup>1</sup>

The statement of the theorem was recently proved in [9] under the stronger assumption that  $f_0$  is close to a round sphere in the  $C^{2,\alpha}$ -topology, using a center manifold analysis which gives related stability results for a couple of other flows; see [2] for an overview. Our method, which is (and has to be) entirely different, involves deriving a priori estimates from the equation satisfied by the curvature, somewhat analogous to the work of Huisken [3, 4]. However, in our problem the crucial estimates are of integral type and the smallness condition is essential in

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<sup>1</sup>*Note added in proof:* a numerical example of a singularity was recently contributed by U. Mayer & G. Simonett: <http://www.math.utah.edu/~mayer/math/numerics.html>.

ruling out possible concentrations related to the scale invariance of the functional.

**2. Estimates for surfaces with locally small Willmore energy**

Here we derive some bounds for immersed surfaces  $f : \Sigma \rightarrow \mathbb{R}^n$  depending on the  $L^2$  norms of their curvature  $A$  and of their Willmore gradient  $\mathbf{W}(f) = \Delta H + Q(A^\circ)H$ , under the assumption that the  $L^2$  norm of  $A^\circ$ , the tracefree part of the curvature, is locally small.

Recall the equations of Mainardi-Codazzi, Gauß and Ricci:

$$(5) \quad (\nabla_X A)(Y, Z) = (\nabla_Y A)(X, Z); \quad \nabla H = -\nabla^* A = -2\nabla^* A^\circ,$$

$$(6) \quad K = \frac{1}{4}|H|^2 - \frac{1}{2}|A^\circ|^2,$$

$$(7) \quad R^\perp(X, Y)\phi = A^\circ(e_i, X)\langle A^\circ(e_i, Y), \phi \rangle - A^\circ(e_i, Y)\langle A^\circ(e_i, X), \phi \rangle.$$

Note  $\langle R^\perp(X, Y)\phi, \phi \rangle = 0$  and in particular  $R^\perp = 0$  for  $n = 3$ , i.e., codimension one. The Codazzi equations imply that  $\nabla A$  and  $\nabla^2 A$  can be expressed by  $\nabla A^\circ$  and  $\nabla^2 A^\circ$ , respectively. In particular one has inequalities

$$(8) \quad |\nabla A| \leq c|\nabla A^\circ|, \quad |\nabla^2 A| \leq c|\nabla^2 A^\circ|.$$

**Lemma 2.1.** *For any  $p$ -linear form  $\phi$  along  $f$  we have*

$$(9) \quad \begin{aligned} & ((\nabla\nabla^* - \nabla^*\nabla)\phi)(X_1, \dots, X_p) \\ &= K\phi(X_1, \dots, X_p) + K \sum_{k=2}^p \phi(X_k, X_2, \dots, X_1, \dots, X_p) \\ &\quad - K \sum_{k=2}^p g(X_1, X_k) \phi(e_i, X_2, \dots, e_i, \dots, X_p) \\ &\quad + R^\perp(e_i, X_1) \phi(e_i, X_2, \dots, X_p) - (\nabla^*T)(X_1, \dots, X_p). \end{aligned}$$

Here the tensor  $T$  is given by

$$T(X_0, X_1, \dots, X_p) = (\nabla_{X_0}\phi)(X_1, X_2, \dots, X_p) - (\nabla_{X_1}\phi)(X_0, X_2, \dots, X_p).$$

*Proof.* From the proof of Lemma 2.1 in [5] we have

$$\begin{aligned} & ((\nabla\nabla^* - \nabla^*\nabla)\phi)(X_1, \dots, X_p) \\ &= (R^p(e_i, X_1)\phi)(e_i, X_2, \dots, X_p) - (\nabla^*T)(X_1, \dots, X_p). \end{aligned}$$

Now the curvature operator  $R^p$  is given by

$$\begin{aligned}
& (R^p(e_i, X_1)\phi)(e_i, X_2, \dots, X_p) \\
&= R^\perp(e_i, X_1)\phi(e_i, X_2, \dots, X_p) \\
&\quad - \phi(R(e_i, X_1)e_i, X_2, \dots, X_p) \\
&\quad - \sum_{k=2}^p \phi(e_i, X_2, \dots, R(e_i, X_1)X_k, \dots, X_p) \\
&= R^\perp(e_i, X_1)\phi(e_i, X_2, \dots, X_p) \\
&\quad - K\phi(g(X_1, e_i)e_i, X_2, \dots, X_p) + K\phi(g(e_i, e_i)X_1, X_2, \dots, X_p) \\
&\quad - K\sum_{k=2}^p \phi(e_i, X_2, \dots, g(X_1, X_k)e_i, \dots, X_p) \\
&\quad + K\sum_{k=2}^p \phi(g(e_i, X_k)e_i, X_2, \dots, X_1, \dots, X_p).
\end{aligned}$$

Inserting yields the desired formula.

q.e.d.

We will need three different choices for  $\phi$  in (9). Taking first  $\phi = A$  yields  $T = 0$  and  $\nabla^* \phi = -\nabla H$  by (5), and we get Simons' identity ([10])

$$\Delta A = \nabla^2 H + 2KA^\circ + R^\perp(e_i, \cdot)A(e_i, \cdot).$$

To bring this in a more useful form, let us denote by  $S^\circ(B)$  the symmetric, tracefree part of any bilinear form with normal values along  $f$ . In particular, we have

$$S^\circ(\nabla^2 H) = \nabla^2 H - \frac{1}{2}g(\cdot, \cdot)\Delta H - \frac{1}{2}R^\perp(\cdot, \cdot)H.$$

Now  $\Delta(\frac{1}{2}g(\cdot, \cdot)H) = \frac{1}{2}g(\cdot, \cdot)\Delta H$  and  $R^\perp(e_i, X)\frac{1}{2}g(e_i, Y)H = -\frac{1}{2}R^\perp(X, Y)H$ , which implies, using (6) and (7),

$$(10) \quad \Delta A^\circ = S^\circ(\nabla^2 H) + \frac{1}{2}|H|^2 A^\circ + A^\circ * A^\circ * A^\circ.$$

Here and in the following we denote by  $A * B$  any universal, linear combination of tensors obtained by tensor product and contraction from  $A$  and  $B$ . Our second choice in (9) is  $\phi = \nabla H$ , where now

$$T(X, Y) = \nabla_{X,Y}^2 H - \nabla_{Y,X}^2 H = R^\perp(X, Y)H.$$

Using again (5), (6) and (7), we infer

$$(11) \quad \nabla^*(\nabla^2 H) = \nabla(\nabla^* \nabla H) - \frac{1}{4}|H|^2 \nabla H + A * A^\circ * \nabla A^\circ.$$

Finally taking  $\phi = \nabla A^\circ$  in (9) yields

$$\begin{aligned} T(X, Y, Z, V) &= (R^2(X, Y) A^\circ)(Z, V) = (A * A * A)(X, Y, Z, V), \\ \nabla^* T &= A * A * \nabla A^\circ. \end{aligned}$$

Thus we obtain from (9) and (6), (7)

$$(12) \quad \nabla^*(\nabla^2 A^\circ) = \nabla(\nabla^* \nabla A^\circ) + A * A * \nabla A^\circ.$$

We now convert (10), (11) and (12) into integral estimates.

**Lemma 2.2.** *If  $f : \Sigma \rightarrow \mathbb{R}^n$  is an immersion with  $\mathbf{W}(f) = W$  and  $\gamma \in C_c^1(\Sigma)$  satisfies  $|\nabla \gamma| \leq \Lambda$ , then*

$$(13) \quad \int |\nabla A|^2 \gamma^2 d\mu \leq \frac{c}{\Lambda^2} \int |W|^2 \gamma^4 d\mu + c \int |A^\circ|^4 \gamma^2 d\mu + c\Lambda^2 \int_{[\gamma>0]} |A|^2 d\mu.$$

*Proof.* Multiply (10) by  $\gamma^2 A^\circ$  and integrate by parts to obtain, after applying (5),

$$\begin{aligned} &\int |\nabla A^\circ|^2 \gamma^2 d\mu + \frac{1}{2} \int |H|^2 |A^\circ|^2 \gamma^2 d\mu \\ &\leq \frac{1}{2} \int |\nabla H|^2 \gamma^2 d\mu + c \int |A^\circ|^4 \gamma^2 d\mu + \int \gamma \nabla \gamma * A^\circ * \nabla A^\circ d\mu. \end{aligned}$$

Using the equation  $\Delta H + Q(A^\circ)H = W$  we have

$$\begin{aligned} \frac{1}{2} \int |\nabla H|^2 \gamma^2 d\mu &= -\frac{1}{2} \int \langle H, \Delta H \rangle \gamma^2 d\mu + \int \gamma \nabla \gamma * A * \nabla A^\circ d\mu \\ &= -\frac{1}{2} \int \langle H, W \rangle \gamma^2 d\mu + \frac{1}{2} \int \langle H, Q(A^\circ)H \rangle \gamma^2 d\mu \\ &\quad + \int \gamma \nabla \gamma * A * \nabla A^\circ d\mu \\ &\leq \frac{c}{\Lambda^2} \int |W|^2 \gamma^4 d\mu + c\Lambda^2 \int_{[\gamma>0]} |H|^2 d\mu \\ &\quad + \frac{1}{2} \int \langle H, Q(A^\circ)H \rangle \gamma^2 d\mu + \int \gamma \nabla \gamma * A * \nabla A^\circ d\mu. \end{aligned}$$

It is easy to see the inequality

$$(14) \quad 0 \leq \langle Q(A^\circ)H, H \rangle \leq |A^\circ|^2 |H|^2.$$

Furthermore we have

$$\int \gamma \nabla \gamma * A * \nabla A^\circ d\mu \leq \frac{1}{2} \int |\nabla A^\circ|^2 \gamma^2 d\mu + c\Lambda^2 \int_{[\gamma>0]} |A|^2 d\mu.$$

Inserting these inequalities, absorbing and recalling (8) proves the claim. q.e.d.

**Lemma 2.3.** *Under the assumptions of Lemma 2.2 we have for  $\eta = \gamma^4$*

$$(15) \quad \begin{aligned} & \int |\nabla^2 H|^2 \eta + \int |A|^2 |\nabla A|^2 \eta d\mu + \int |A|^4 |A^\circ|^2 \eta d\mu \\ & \leq c \int |W|^2 \eta d\mu + c \int (|A^\circ|^2 |\nabla A^\circ|^2 + |A^\circ|^6) \eta d\mu \\ & \quad + c\Lambda^4 \int_{[\gamma>0]} |A|^2 d\mu. \end{aligned}$$

*Proof.* We start multiplying (11) by  $\eta \nabla H$  and integrating by parts. This yields

$$\begin{aligned} & \int |\nabla^2 H|^2 \eta d\mu + \frac{1}{4} \int |H|^2 |\nabla H|^2 \eta d\mu \\ & \leq \int |\Delta H|^2 \eta d\mu + \int A * A^\circ * \nabla A^\circ * \nabla A^\circ \eta d\mu \\ & \quad + \int \gamma^3 \nabla \gamma * \nabla H * \nabla^2 H d\mu \\ & \leq c \int |W|^2 \eta d\mu + c \int |A^\circ|^4 |H|^2 \eta d\mu \\ & \quad + \varepsilon \int |H|^2 |\nabla A^\circ|^2 \eta d\mu + c(\varepsilon) \int |A^\circ|^2 |\nabla A^\circ|^2 \eta d\mu \\ & \quad + \frac{1}{2} \int |\nabla^2 H|^2 \eta d\mu + c\Lambda^2 \int |\nabla H|^2 \gamma^2 d\mu. \end{aligned}$$

Now by (13) we can estimate

$$\int |\nabla H|^2 \gamma^2 d\mu \leq \frac{c}{\Lambda^2} \int |W|^2 \eta d\mu + \frac{c}{\Lambda^2} \int |A^\circ|^6 \eta d\mu + c\Lambda^2 \int_{[\gamma>0]} |A|^2 d\mu.$$

Using the inequality  $c|A^\circ|^4|H|^2 \leq \varepsilon|H|^4|A^\circ|^2 + c(\varepsilon)|A^\circ|^6$  and rearranging, we arrive at

$$\begin{aligned}
(16) \quad & \int |\nabla^2 H|^2 \eta \, d\mu + \int |H|^2 |\nabla H|^2 \eta \, d\mu \\
& \leq c \int |W|^2 \eta \, d\mu + c\Lambda^4 \int_{[\gamma>0]} |A|^2 \, d\mu \\
& \quad + c(\varepsilon) \int |A^\circ|^2 |\nabla A^\circ|^2 \eta \, d\mu + c(\varepsilon) \int |A^\circ|^6 \eta \, d\mu \\
& \quad + \varepsilon \int |H|^4 |A^\circ|^2 \eta \, d\mu + \varepsilon \int |H|^2 |\nabla A^\circ|^2 \eta \, d\mu.
\end{aligned}$$

Next we use (10) to compute

$$\begin{aligned}
& \int |H|^2 |\nabla A^\circ|^2 \eta \, d\mu \\
& = - \int |H|^2 \langle A^\circ, \Delta A^\circ \rangle \eta \, d\mu + \int H * \nabla H * A^\circ * \nabla A^\circ \eta \, d\mu \\
& \quad + \int |H|^2 A^\circ * \nabla A^\circ * \nabla \eta \, d\mu \\
& = - \int |H|^2 \langle A^\circ, \nabla^2 H + \frac{1}{2}|H|^2 A^\circ + A^\circ * A^\circ * A^\circ \rangle \eta \, d\mu \\
& \quad + \int H * \nabla H * A^\circ * \nabla A^\circ \eta \, d\mu + \int |H|^2 A^\circ * \nabla A^\circ * \nabla \eta \, d\mu \\
& \leq \frac{1}{2} \int |H|^2 |\nabla H|^2 \eta \, d\mu + \int H * \nabla H * A^\circ * \nabla A^\circ \eta \, d\mu \\
& \quad + \int |H|^2 A^\circ * \nabla A^\circ * \gamma^3 \nabla \gamma \, d\mu \\
& \quad - \frac{1}{2} \int |H|^4 |A^\circ|^2 \eta \, d\mu + c \int |H|^2 |A^\circ|^4 \eta \, d\mu \\
& \leq \left( \frac{1}{2} + \delta \right) \int |H|^2 |\nabla H|^2 \eta \, d\mu + c(\delta) \int |A^\circ|^2 |\nabla A^\circ|^2 \eta \, d\mu \\
& \quad + \delta \int |H|^2 |\nabla A^\circ|^2 \eta \, d\mu + c(\delta) \Lambda^2 \int |H|^2 |A^\circ|^2 \gamma^2 \, d\mu \\
& \quad - \frac{1}{2} \int |H|^4 |A^\circ|^2 \eta \, d\mu + c \int |H|^2 |A^\circ|^4 \eta \, d\mu.
\end{aligned}$$

From the inequalities

$$\begin{aligned} c \int |H|^2 |A^\circ|^4 \eta d\mu &\leq \delta \int |H|^4 |A^\circ|^2 \eta d\mu + c(\delta) \int |A^\circ|^6 \eta d\mu, \\ c(\delta) \Lambda^2 \int |H|^2 |A^\circ|^2 \gamma^2 d\mu &\leq \delta \int |H|^4 |A^\circ|^2 \eta d\mu + c(\delta) \Lambda^4 \int_{[\gamma>0]} |A^\circ|^2 d\mu, \end{aligned}$$

we see that

$$\begin{aligned} (17) \quad &(1 - \delta) \int |H|^2 |\nabla A^\circ|^2 \eta d\mu + \left(\frac{1}{2} - 2\delta\right) \int |H|^4 |A^\circ|^2 \eta d\mu \\ &\leq \left(\frac{1}{2} + \delta\right) \int |H|^2 |\nabla H|^2 \eta d\mu \\ &\quad + c(\delta) \left( \int |A^\circ| |\nabla A^\circ|^2 \eta d\mu + \int |A^\circ|^6 \eta d\mu + \Lambda^4 \int_{[\gamma>0]} |A|^2 d\mu \right). \end{aligned}$$

Adding the inequalities (16) and (17) yields

$$\begin{aligned} &\int |\nabla^2 H|^2 \eta d\mu + \left(\frac{1}{2} - \delta\right) \int |H|^2 |\nabla H|^2 \eta d\mu \\ &\quad + (1 - \delta - \varepsilon) \int |H|^2 |\nabla A^\circ|^2 \eta d\mu + \left(\frac{1}{2} - 2\delta - \varepsilon\right) \int |H|^4 |A^\circ|^2 \eta d\mu \\ &\leq c(\delta, \varepsilon) \left( \int |A^\circ|^2 |\nabla A^\circ|^2 \eta d\mu + \int |A^\circ|^6 \eta d\mu + \Lambda^4 \int_{[\gamma>0]} |A|^2 d\mu \right) \\ &\quad + c \int |W|^2 \eta d\mu. \end{aligned}$$

The claim of the Lemma follows by choosing  $\varepsilon = \delta = \frac{1}{8}$ . q.e.d.

**Proposition 2.4.** *If  $f : \Sigma \rightarrow \mathbb{R}^n$  is an immersion with  $\mathbf{W}(f) = W$  and  $\eta = \gamma^4$ , where  $\gamma \in C_c^1(\Sigma)$  satisfies  $|\nabla \gamma| \leq \Lambda$ , then*

$$\begin{aligned} &\int |\nabla^2 A|^2 \eta d\mu + \int |A|^2 |\nabla A|^2 \eta d\mu + \int |A|^4 |A^\circ|^2 \eta d\mu \\ &\leq c \int |W|^2 \eta d\mu + c \Lambda^4 \int_{[\gamma>0]} |A|^2 d\mu + c \int (|A^\circ|^2 |\nabla A^\circ|^2 + |A^\circ|^6) \eta d\mu. \end{aligned}$$



*Proof.* Multiply (12) by  $\eta \nabla A^\circ$ , integrate by parts and apply (10) to get

$$\begin{aligned}
\int |\nabla^2 A^\circ|^2 \eta d\mu &\leq \int |\Delta A^\circ|^2 \eta d\mu + c \int |A|^2 |\nabla A|^2 \eta d\mu \\
&\quad + \int \gamma^3 \nabla \gamma * \nabla A^\circ * \nabla^2 A^\circ d\mu \\
&\leq c \int |\nabla^2 H|^2 \eta d\mu \\
&\quad + c \int |A|^2 |\nabla A|^2 d\mu + c \int |A|^4 |A^\circ|^2 \eta d\mu \\
&\quad + \frac{1}{2} \int |\nabla^2 A^\circ|^2 \eta d\mu + c \Lambda^2 \int |\nabla A^\circ|^2 \gamma^2 d\mu.
\end{aligned}$$

The claim now follows from Lemma 2.2 and Lemma 2.3, recalling (8).

q.e.d.

We next need a multiplicative Sobolev inequality.

**Lemma 2.5.** *Under the assumptions of Proposition 2.4 we have*

$$\begin{aligned}
(18) \quad &\int (|A^\circ|^2 |\nabla A^\circ|^2 + |A^\circ|^6) \eta d\mu \\
&\leq c \int_{[\gamma>0]} |A^\circ|^2 d\mu \cdot \int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) \eta d\mu \\
&\quad + c \Lambda^4 \left( \int_{[\gamma>0]} |A^\circ|^2 d\mu \right)^2.
\end{aligned}$$

*Proof.* Recall the Michael-Simon Sobolev inequality ([7])

$$(19) \quad \left( \int_{\Sigma} u^2 d\mu \right)^{\frac{1}{2}} \leq c \left( \int_{\Sigma} |\nabla u| d\mu + \int_{\Sigma} |H| |u| d\mu \right),$$

with  $c = c(n)$ . Letting  $u = |A^\circ| |\nabla A^\circ| \gamma^2$  we obtain

$$\begin{aligned}
& \int |A^\circ|^2 |\nabla A^\circ|^2 \eta \, d\mu \\
& \leq c \left( \int |A^\circ| |\nabla^2 A^\circ| \gamma^2 \, d\mu \right)^2 + c \left( \int |\nabla A^\circ|^2 \gamma^2 \, d\mu \right)^2 \\
& \quad + c \left( \int |A| |A^\circ| |\nabla A^\circ| \gamma^2 \, d\mu \right)^2 + c \left( \int |A^\circ| |\nabla A^\circ| \gamma |\nabla \gamma| \, d\mu \right)^2 \\
& \leq c \int_{[\gamma>0]} |A^\circ|^2 \, d\mu \int (|\nabla^2 A^\circ|^2 + |A|^2 |\nabla A^\circ|^2) \eta \, d\mu \\
& \quad + c \Lambda^4 \left( \int_{[\gamma>0]} |A^\circ|^2 \, d\mu \right)^2 + c \left( \int |\nabla A^\circ|^2 \gamma^2 \, d\mu \right)^2.
\end{aligned}$$

In the last term, we integrate by parts to get

$$\begin{aligned}
(20) \quad \int |\nabla A^\circ|^2 \gamma^2 \, d\mu & \leq c \int |A^\circ| |\nabla^2 A^\circ| \gamma^2 \, d\mu + c \Lambda \int |A^\circ| |\nabla A^\circ| \gamma \, d\mu \\
& \leq c \left( \int_{[\gamma>0]} |A^\circ|^2 \, d\mu \right)^{\frac{1}{2}} \cdot \left( \int |\nabla^2 A^\circ|^2 \eta \, d\mu \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \int |\nabla A^\circ|^2 \gamma^2 \, d\mu + c \Lambda^2 \int_{[\gamma>0]} |A^\circ|^2 \, d\mu.
\end{aligned}$$

Absorbing and inserting proves the claimed inequality for the first term in (18). For the other term, choose  $u = |A^\circ|^3 \gamma^2$  in (19) and compute

$$\begin{aligned}
& \int |A^\circ|^6 \eta \, d\mu \\
& \leq c \left( \int |A^\circ|^2 |\nabla A^\circ| \gamma^2 \, d\mu + \int |A| |A^\circ|^3 \gamma^2 \, d\mu + c \Lambda \int |A^\circ|^3 \gamma \, d\mu \right)^2 \\
& \leq c \left( \int |\nabla A^\circ|^2 \gamma^2 \, d\mu \right)^2 + c \int_{[\gamma>0]} |A^\circ|^2 \, d\mu \cdot \int |A|^2 |A^\circ|^4 \eta \, d\mu \\
& \quad + c \Lambda^4 \left( \int_{[\gamma>0]} |A^\circ|^2 \, d\mu \right)^2.
\end{aligned}$$

Combining with (20) proves the estimate for the second term on the left of (18). q.e.d.

**Proposition 2.6.** *Let  $f : \Sigma \rightarrow \mathbb{R}^n$  be an immersed surface, and let  $\Lambda = \|\nabla\gamma\|_{L^\infty}$ , where  $\gamma$  has compact support on  $\Sigma$ . There exists a constant  $\varepsilon_0 = \varepsilon_0(n) > 0$  such that if*

$$\int_{[\gamma>0]} |A^\circ|^2 d\mu < \varepsilon_0,$$

then we have for a constant  $c = c(n) < \infty$

$$\begin{aligned} & \int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) \gamma^4 d\mu \\ & \leq c \int |\mathbf{W}(f)|^2 \gamma^4 d\mu + c \Lambda^4 \int_{[\gamma>0]} |A|^2 d\mu. \end{aligned}$$

This is an immediate consequence of Proposition 2.4 and Lemma 2.5. As a first application we deduce the following result.

**Theorem 2.7** (Gap Lemma). *Let  $f : \Sigma \rightarrow \mathbb{R}^n$  be a properly immersed (compact or noncompact) Willmore surface, and let  $\Sigma_\varrho(0) = f^{-1}(B_\varrho(0))$ . If*

$$\begin{aligned} \liminf_{\varrho \rightarrow \infty} \frac{1}{\varrho^4} \int_{\Sigma_\varrho(0)} |A|^2 d\mu &= 0, \quad \text{and} \\ \int_{\Sigma} |A^\circ|^2 d\mu < \varepsilon_0 &= \varepsilon_0(n), \end{aligned}$$

then  $f$  is an embedded plane or sphere.

*Proof.* We take  $\gamma(p) = \varphi(\frac{1}{\varrho} |f(p)|)$ , where  $\varphi \in C^1(\mathbb{R})$  satisfies  $\varphi(s) = 1$  for  $s \leq \frac{1}{2}$ ,  $\varphi(s) = 0$  for  $s \geq 1$  and  $\varphi \geq 0$ . Then we have  $\Lambda = c/\varrho$  in Proposition 2.6. Since  $\mathbf{W}(f) = 0$  by assumption, we can let  $\varrho \rightarrow \infty$  and conclude  $A^\circ \equiv 0$ . This implies, by a standard result of differential geometry [13], that  $f$  maps into a fixed, round 2-sphere or plane  $S \subset \mathbb{R}^n$ . As  $f$  is complete, it follows that  $f : (\Sigma, g) \rightarrow S$  is a global isometry. q.e.d.

We shall now derive an  $L^\infty$  bound for  $A^\circ$  from Proposition 2.6.

**Lemma 2.8.** *For  $\gamma \in C_c^1(\Sigma)$  with  $|\nabla\gamma| \leq \Lambda$  and any normal  $p$ -form  $\phi$  along  $f$  we have the inequality*

$$\|\gamma^2 \phi\|_{L^\infty}^4 \leq c \|\gamma^2 \phi\|_{L^2}^2 \left[ \int (|\nabla^2 \phi|^2 + |H|^4 |\phi|^2) \gamma^4 d\mu + \Lambda^4 \int_{[\gamma>0]} |\phi|^2 d\mu \right].$$

*Proof.* This is Lemma 4.3 in [5], except that there a bound on the second derivatives of  $\gamma$  was assumed. Letting  $\psi = \gamma^2\phi$  we apply Theorem 5.6 in [5] to obtain

$$(21) \quad \begin{aligned} \|\psi\|_{L^\infty}^6 &\leq c \|\psi\|_{L^2}^2 (\|\nabla\psi\|_{L^4}^4 + \|H\psi\|_{L^4}^4) \\ &\leq c \|\psi\|_{L^2}^2 \left( \int \gamma^8 |\nabla\phi|^4 d\mu + \Lambda^4 \int \gamma^4 |\phi|^4 d\mu + \int |H|^4 |\psi|^4 d\mu \right). \end{aligned}$$

The three integrals on the right are estimated as follows (starting with the third):

$$(22) \quad \int |H|^4 |\psi|^4 d\mu \leq \|\psi\|_{L^\infty}^2 \int |H|^4 |\phi|^2 \gamma^4 d\mu,$$

$$(23) \quad \Lambda^4 \int \gamma^4 |\phi|^4 d\mu \leq \|\psi\|_{L^\infty}^2 \Lambda^4 \int_{[\gamma>0]} |\phi|^2 d\mu.$$

By partial integration, we infer

$$\begin{aligned} \int |\nabla\phi|^2 \gamma^2 d\mu &\leq c \int |\phi| |\nabla^2\phi| \gamma^2 d\mu + c\Lambda \int |\phi| |\nabla\phi| \gamma d\mu \\ &\leq \frac{c}{\Lambda^2} \int |\nabla^2\phi|^2 \gamma^4 d\mu + c\Lambda^2 \int_{[\gamma>0]} |\phi|^2 d\mu + \frac{1}{2} \int |\nabla\phi|^2 \gamma^2 d\mu. \end{aligned}$$

Using again integration by parts and Cauchy-Schwarz

$$\begin{aligned} \int |\nabla\phi|^4 \gamma^8 d\mu &\leq c \left( \int |\phi| |\nabla\phi|^2 |\nabla^2\phi| \gamma^8 d\mu + \Lambda \int |\phi| |\nabla\phi|^3 \gamma^7 d\mu \right) \\ &\leq c \|\psi\|_{L^\infty} \left( \int |\nabla\phi|^4 \gamma^8 d\mu \right)^{\frac{1}{2}} \left( \int |\nabla^2\phi|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} \\ &\quad + c\Lambda \|\psi\|_{L^\infty} \left( \int |\nabla\phi|^4 \gamma^8 d\mu \right)^{\frac{1}{2}} \left( \int |\nabla\phi|^2 \gamma^2 d\mu \right)^{\frac{1}{2}}. \end{aligned}$$

Combining the last two inequalities, we get

$$(24) \quad \int |\nabla\phi|^4 \gamma^8 d\mu \leq c \|\psi\|_{L^\infty}^2 \left( \int |\nabla^2\phi|^2 \gamma^4 d\mu + c\Lambda^4 \int_{[\gamma>0]} |\phi|^2 d\mu \right).$$

Inserting (22)–(24) into (21) proves the claim. q.e.d.

Combining Proposition 2.6 and Lemma 2.8, where  $\phi = A^\circ$  and  $\gamma$  is a cutoff function depending on extrinsic distance as in Theorem 2.7, we obtain the following “partial” curvature estimate.

**Theorem 2.9** (Tracefree Curvature Estimate). *Let  $f : \Sigma \rightarrow \mathbb{R}^n$  be an immersed surface with  $\Sigma_\varrho = f^{-1}(B_\varrho(x_0)) \subset\subset \Sigma$ , and suppose that*

$$\int_{\Sigma_\varrho} |A^\circ|^2 d\mu < \varepsilon_0,$$

where  $\varepsilon_0 = \varepsilon_0(n) > 0$  is a fixed constant. Then

$$(25) \quad \|A^\circ\|_{L^\infty(\Sigma_{\varrho/2})}^2 \leq c \left( \|\mathbf{W}(f)\|_{L^2(\Sigma_\varrho)} + \frac{1}{\varrho^2} \|A\|_{L^2(\Sigma_\varrho)} \right) \|A^\circ\|_{L^2(\Sigma_\varrho)}.$$

Assuming smallness of the full second fundamental form  $A$ , one easily adapts the arguments above to also prove the following:

**Theorem 2.10** (Curvature Estimate). *Let  $f : \Sigma \rightarrow \mathbb{R}^n$  be an immersed surface,  $\Sigma_\varrho = f^{-1}(B_\varrho(x_0)) \subset\subset \Sigma$  and suppose*

$$\int_{\Sigma_\varrho} |A|^2 d\mu < \varepsilon_0,$$

where  $\varepsilon_0 = \varepsilon_0(n)$  is a fixed constant. Then

$$(26) \quad \|A\|_{L^\infty(\Sigma_{\varrho/2})}^2 \leq c \left( \|\mathbf{W}(f)\|_{L^2(\Sigma_\varrho)} + \frac{1}{\varrho^2} \|A\|_{L^2(\Sigma_\varrho)} \right) \|A\|_{L^2(\Sigma_\varrho)}.$$

**Remark 2.11.** The statements of the Theorems 2.7, 2.9 and 2.10 clearly also hold with the extrinsic distance sets  $\Sigma_\varrho(x_0)$  replaced by distance sets with respect to the *intrinsic* distance function, since only a bound on the first derivatives of the cutoff function was needed.

### 3. Local estimates for the flow

We now consider solutions  $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^n$  to the gradient flow for the Willmore integral,

$$\partial_t f = -\mathbf{W}(f).$$

We abbreviate  $\mathbf{W}(f) =: W$  in the following and compute first a precise

formula for the evolution of the energy density. Recall from [5]:

$$(27) \quad \partial_t(d\mu) = \langle H, W \rangle d\mu,$$

$$(28) \quad \partial_t^\perp H = - \left( \Delta W + Q(A^\circ)W + \frac{1}{2}H\langle H, W \rangle \right),$$

$$(29) \quad \begin{aligned} \partial_t^\perp A(X, Y) = & -\nabla_{X, Y}^2 W + \frac{1}{2}g(X, Y) \left[ Q(A^\circ)W + \frac{1}{2}H\langle H, W \rangle \right] \\ & + \frac{1}{2}H\langle A^\circ(X, Y), W \rangle + \frac{1}{2}A^\circ(X, Y)\langle H, W \rangle \\ & + \frac{1}{2}R^\perp(X, Y)W. \end{aligned}$$

Here we used (2.18), (2.6) and (2.3) from [5]. Furthermore, using (2.15) in [5] we infer

$$\begin{aligned} \partial_t^\perp \left( \frac{1}{2}g(X, Y)H \right) = & -\frac{1}{2}g(X, Y) \left( \Delta W + Q(A^\circ)W + \frac{1}{2}H\langle H, W \rangle \right) \\ & + \langle A^\circ(X, Y), W \rangle H + \frac{1}{2}g(X, Y)H\langle H, W \rangle, \end{aligned}$$

and subtracting this from (29) yields

$$(30) \quad \begin{aligned} \partial_t^\perp A^\circ(X, Y) = & -S^\circ(\nabla^2 W) + g(X, Y)Q(A^\circ)W \\ & + \frac{1}{2}A^\circ(X, Y)\langle H, W \rangle - \frac{1}{2}H\langle A^\circ(X, Y), W \rangle. \end{aligned}$$

Recall that  $S^\circ(\dots)$  denotes the symmetric, tracefree component. We compute separately for  $H$  and  $A^\circ$ . By (27) and (28)

$$\begin{aligned} & \partial_t \left( \frac{1}{2}|H|^2 d\mu \right) \\ & = - \left\langle \Delta W + Q(A^\circ)W + \frac{1}{2}H\langle H, W \rangle, H \right\rangle d\mu + \frac{1}{2}|H|^2 \langle H, W \rangle d\mu \\ & = -\langle \Delta H + Q(A^\circ)H, W \rangle d\mu + (\langle \Delta H, W \rangle - \langle H, \Delta W \rangle) d\mu \\ & = -|W|^2 d\mu + \nabla_{e_i}(\langle \nabla_{e_i} H, W \rangle - \langle H, \nabla_{e_i} W \rangle) d\mu, \end{aligned}$$

whence

$$(31) \quad \partial_t \left( \frac{1}{2}|H|^2 d\mu \right) + |W|^2 d\mu = (\nabla^* \alpha) d\mu,$$

where  $\alpha$  is the 1-form given by

$$(32) \quad \alpha(X) = \nabla_X \langle H, W \rangle - 2\langle \nabla_X H, W \rangle.$$

In order to compute for  $A^\circ$ , we first have (using again (2.15) in [5]) for a  $g$ -orthonormal basis

$$\begin{aligned} g(\partial_t e_i, e_j) + g(e_i, \partial_t e_j) &= \partial_t (g(e_i, e_j)) - (\partial_t g)(e_i, e_j) \\ &= -2\langle A(e_i, e_j), W \rangle \\ &= -2\langle A^\circ(e_i, e_j), W \rangle - \delta_{ij}\langle H, W \rangle. \end{aligned}$$

This implies further

$$\begin{aligned} &\langle A^\circ(\partial_t e_i, e_k), A^\circ(e_i, e_k) \rangle \\ &= g(\partial_t e_i, e_j)\langle A^\circ(e_j, e_k), A^\circ(e_i, e_k) \rangle \\ &= -\left( \langle A^\circ(e_i, e_j), W \rangle + \frac{1}{2}\delta_{ij}\langle H, W \rangle \right) \langle A^\circ(e_i, e_k), A^\circ(e_j, e_k) \rangle \\ &= -\langle A^\circ(e_i, e_k)\langle A^\circ(e_i, e_j), W \rangle, A^\circ(e_j, e_k) \rangle - \frac{1}{2}|A^\circ|^2\langle H, W \rangle \\ &= -\left\langle \frac{1}{2}g(e_j, e_k)Q(A^\circ)W, A^\circ(e_j, e_k) \right\rangle - \frac{1}{2}|A^\circ|^2\langle H, W \rangle \\ &= -\frac{1}{2}|A^\circ|^2\langle H, W \rangle, \end{aligned}$$

where we used (2.5) from [5]. We use this and (30) to compute

$$\begin{aligned} &\partial_t (|A^\circ|^2 d\mu) \\ &= 2\langle (\partial_t A^\circ)(e_i, e_k), A^\circ(e_i, e_k) \rangle d\mu \\ &\quad + 2\langle A^\circ(\partial_t e_i, e_k) + A^\circ(e_i, \partial_t e_k), A^\circ(e_i, e_k) \rangle d\mu + |A^\circ|^2 \langle H, W \rangle d\mu \\ &= -2\langle \nabla^2 W, A^\circ \rangle d\mu + |A^\circ|^2 \langle H, W \rangle d\mu \\ &\quad - \langle A^\circ(e_i, e_k), W \rangle \langle A^\circ(e_i, e_k), H \rangle d\mu \\ &\quad - 2|A^\circ|^2 \langle H, W \rangle d\mu + |A^\circ|^2 \langle H, W \rangle d\mu \\ &= -2\langle \nabla^2 W, A^\circ \rangle d\mu - \langle Q(A^\circ)H, W \rangle d\mu \\ &= (-2\nabla_{e_i}\langle \nabla_{e_j} W, A^\circ(e_i, e_j) \rangle + \langle \nabla_{e_j} W, \nabla_{e_j} H \rangle) d\mu \\ &\quad - \langle Q(A^\circ)H, W \rangle d\mu \\ &= (-2\nabla_{e_i}\langle \nabla_{e_j} W, A^\circ(e_i, e_j) \rangle + \nabla_{e_i}\langle W, \nabla_{e_i} H \rangle) d\mu \\ &\quad - \langle W, \Delta H \rangle d\mu - \langle Q(A^\circ)H, W \rangle d\mu. \end{aligned}$$

Thus we have shown

$$(33) \quad \partial_t (|A^\circ|^2 d\mu) + |W|^2 d\mu = (\nabla^* \beta) d\mu,$$

where  $\beta$  is the 1-form defined by

$$(34) \quad \beta(X) = 2\langle \nabla_{e_j} W, A^\circ(X, e_j) \rangle - \langle \nabla_X H, W \rangle.$$

**Lemma 3.1.** *If  $f$  is a Willmore flow, then for any function  $\eta$  and  $W = \mathbf{W}(f)$  we have:*

$$(35) \quad \begin{aligned} \partial_t \int \frac{1}{2} |H|^2 \eta \, d\mu + \int |W|^2 \eta \, d\mu \\ = \int \left( \frac{1}{2} |H|^2 \partial_t \eta - \langle H, W \rangle \Delta \eta - 2 \langle \nabla_{\text{grad } \eta} H, W \rangle \right) d\mu, \end{aligned}$$

$$(36) \quad \begin{aligned} \partial_t \int |A^\circ|^2 \eta \, d\mu + \int |W|^2 \eta \, d\mu \\ = \int \left( |A^\circ|^2 \partial_t \eta - 2 \langle A^\circ(e_i, e_j), W \rangle \nabla_{e_i, e_j}^2 \eta - 2 \langle \nabla_{\text{grad } \eta} H, W \rangle \right) d\mu. \end{aligned}$$

*Proof.* Formula (35) is immediate from (31) and (32). For (36) we compute for  $\beta$  as in (34):

$$\begin{aligned} \int \eta \nabla^* \beta \, d\mu &= \int \left( 2 \langle \nabla_{e_j} W, A^\circ(\text{grad } \eta, e_j) \rangle - \langle \nabla_{\text{grad } \eta} H, W \rangle \right) d\mu \\ &= - \int 2 \langle (\nabla_{e_j} A^\circ)(e_j, \text{grad } \eta), W \rangle d\mu \\ &\quad - \int 2 \langle A^\circ(\nabla_{e_j} \text{grad } \eta, e_j), W \rangle d\mu - \int \langle \nabla_{\text{grad } \eta} H, W \rangle d\mu \\ &= - \int 2 \nabla_{e_i, e_j}^2 \eta \langle A^\circ(e_i, e_j), W \rangle d\mu - \int 2 \langle \nabla_{\text{grad } \eta} H, W \rangle d\mu, \end{aligned}$$

which, together with (33), proves (36). q.e.d.

In controlling the energy density in time, difficulties arise because of the dependence of  $\partial_t \eta$  and  $\nabla^2 \eta$  on  $f$ , and since  $\mathbf{W}(f)$  differs from  $\Delta H$  by the term  $Q(A^\circ)H$ . For a ball  $B_\varrho = B_\varrho(x_0) \subset \mathbb{R}^n$  and  $f : \Sigma \rightarrow \mathbb{R}^n$  we adopt as in Section 2 the notation

$$\Sigma_\varrho(x_0) = f^{-1}(B_\varrho(x_0))$$

and consider a cutoff function  $\tilde{\gamma} \in C_c^1(B_\varrho)$ ,  $\tilde{\gamma} \geq 0$ , such that

$$(37) \quad |D \tilde{\gamma}| \leq \frac{c}{\varrho}, \quad |D^2 \tilde{\gamma}| \leq \frac{c}{\varrho^2}.$$

We put  $\gamma = \tilde{\gamma} \circ f$  and observe

$$(38) \quad \begin{aligned} \nabla \gamma &= (D \tilde{\gamma} \circ f) \cdot Df \\ \nabla^2 \gamma &= (D^2 \tilde{\gamma} \circ f)(Df, Df) + (D \tilde{\gamma} \circ f) \cdot A(\cdot, \cdot). \end{aligned}$$



**Lemma 3.2.** *If  $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^n$  is a Willmore flow and  $\eta = \gamma^4$  for  $\gamma = \tilde{\gamma} \circ f$  with  $\tilde{\gamma}$  as in (37), then we have for  $W = \mathbf{W}(f)$*

$$(39) \quad \begin{aligned} \partial_t \int |A^\circ|^2 \eta \, d\mu + \frac{1}{2} \int |W|^2 \eta \, d\mu \\ \leq \frac{c}{\varrho^2} \int |A|^2 |A^\circ|^2 \gamma^2 \, d\mu + \frac{c}{\varrho^4} \int_{[\gamma>0]} |A|^2 \, d\mu \end{aligned}$$

$$(40) \quad \begin{aligned} \partial_t \int |A|^2 \eta \, d\mu + \int |W|^2 \eta \, d\mu \\ \leq \frac{c}{\varrho^2} \int |A|^4 \gamma^2 \, d\mu + \frac{c}{\varrho^4} \int_{[\gamma>0]} |A|^2 \, d\mu. \end{aligned}$$

*Proof.* We estimate the terms in (36) and (35). We have

$$\begin{aligned} \int \gamma^2 |\nabla H|^2 \, d\mu &= - \int \gamma^2 \langle H, \Delta H \rangle \, d\mu + \frac{c}{\varrho} \int \gamma |H| |\nabla H| \, d\mu \\ &\leq - \int \gamma^2 \langle H, W \rangle \, d\mu + c \int \gamma^2 |A^\circ|^2 |H|^2 \, d\mu \\ &\quad + \frac{1}{2} \int \gamma^2 |\nabla H|^2 \, d\mu + \frac{c}{\varrho^2} \int_{[\gamma>0]} |H|^2 \, d\mu. \end{aligned}$$

As

$$\int |\nabla \eta| |\nabla H| |W| \, d\mu \leq \varepsilon \int |W|^2 \eta \, d\mu + \frac{c(\varepsilon)}{\varrho^2} \int \gamma^2 |\nabla H|^2 \, d\mu,$$

we obtain by combining

$$\begin{aligned} - \int 2 \langle \nabla_{\text{grad } \eta} H, W \rangle \, d\mu &\leq \varepsilon \int |W|^2 \eta \, d\mu + \frac{c(\varepsilon)}{\varrho^2} \int \gamma^2 |A^\circ|^2 |H|^2 \, d\mu \\ &\quad + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |H|^2 \, d\mu. \end{aligned}$$

Next using (38)

$$\begin{aligned} - \int 2 \langle A^\circ(e_i, e_j), W \rangle \nabla_{e_i, e_j}^2 \eta \, d\mu &\leq c \int |A^\circ| |W| \left( \frac{1}{\varrho^2} \gamma^2 + \frac{1}{\varrho} \gamma^3 |A^\circ| \right) \, d\mu \\ &\leq \varepsilon \int |W|^2 \eta \, d\mu + \frac{c(\varepsilon)}{\varrho^2} \int |A^\circ|^4 \gamma^2 \, d\mu \\ &\quad + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A^\circ|^2 \, d\mu. \end{aligned}$$

Finally

$$\begin{aligned} \int |A^\circ|^2 \partial_t \eta \, d\mu &\leq \frac{c}{\varrho} \int |A^\circ|^2 |W| \gamma^3 \, d\mu \\ &\leq \varepsilon \int |W|^2 \gamma^4 \, d\mu + \frac{c(\varepsilon)}{\varrho^2} \int |A^\circ|^4 \gamma^2 \, d\mu. \end{aligned}$$

Combining the three estimates and absorbing for  $\varepsilon > 0$  small, we obtain (39). The estimate (40) follows analogously from (35). q.e.d.

**Lemma 3.3.** *Let  $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^n$  be a Willmore flow. If*

$$(41) \quad \int_{\Sigma_\varrho(x_0)} |A^\circ|^2 \, d\mu < \varepsilon_0 \quad \text{at some time } t \in [0, T),$$

then for a constant  $c_0 > 0$  we have at time  $t$

$$(42) \quad \begin{aligned} \partial_t \int |A^\circ|^2 \gamma^4 \, d\mu + c_0 \int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) \gamma^4 \, d\mu \\ \leq \frac{c}{\varrho^4} \int_{\Sigma_\varrho(x_0)} |A|^2 \, d\mu, \end{aligned}$$

and

$$(43) \quad \begin{aligned} \partial_t \int |H|^2 \gamma^4 \, d\mu + c_0 \int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) \gamma^4 \, d\mu \\ \leq \frac{c}{\varrho^4} \int_{\Sigma_\varrho(x_0)} |A|^2 \, d\mu. \end{aligned}$$

*Proof.* (42) follows by combining (39) with Proposition 2.6, after estimating

$$\frac{c}{\varrho^2} \int |A|^2 |A^\circ|^2 \gamma^2 \, d\mu \leq \varepsilon \int |A|^4 |A^\circ|^2 \gamma^4 \, d\mu + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma > 0]} |A^\circ|^2 \, d\mu.$$

For the other bound we must go back to (35), estimating the three terms on the right hand side. We have

$$\int \frac{1}{2} |H|^2 \partial_t \eta \, d\mu = - \int \frac{1}{2} |H|^2 D \tilde{\eta} \circ f \cdot (\Delta H + Q(A^\circ) H) \, d\mu.$$

By Young's inequality with 4 and 4/3, we have

$$(44) \quad \frac{1}{\varrho} \int |A^\circ|^2 |A|^3 \gamma^3 d\mu \leq \varepsilon \int |A^\circ|^{8/3} |A|^{10/3} \gamma^4 d\mu + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A|^2 d\mu.$$

Using integration by parts, we infer

$$\begin{aligned} & \int H * H \langle D\tilde{\eta} \circ f, \Delta H \rangle d\mu \\ & \leq \frac{c}{\varrho} \int |H| |\nabla H|^2 \gamma^3 d\mu + \frac{c}{\varrho^2} \int |H|^2 |\nabla H| \gamma^2 d\mu \\ & \leq \varepsilon \int |H|^2 |\nabla H|^2 \gamma^4 d\mu + \frac{c(\varepsilon)}{\varrho^2} \int |\nabla H|^2 \gamma^2 d\mu \\ & \quad + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |H|^2 d\mu. \end{aligned}$$

In the proof of Lemma 3.2, we have already shown by partial integration that

$$\begin{aligned} \int \gamma^2 |\nabla H|^2 d\mu & \leq \delta \varrho^2 \int |W|^2 \gamma^4 d\mu + \frac{c(\delta)}{\varrho^2} \int_{[\gamma>0]} |A|^2 d\mu \\ & \quad + \delta \varrho^2 \int |A|^2 |A^\circ|^4 \gamma^4 d\mu, \end{aligned}$$

so that by combining we obtain

$$(45) \quad \begin{aligned} & \int H * H \langle D\tilde{\eta} \circ f, \Delta H \rangle d\mu \\ & \leq \varepsilon \int (|W|^2 + |A|^2 |\nabla A|^2 + |A|^2 |A^\circ|^4) \gamma^4 d\mu \\ & \quad + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A|^2 d\mu. \end{aligned}$$

Thus (44) and (45) estimate the first of the three terms on the right hand side of (35). For the second we use

$$\begin{aligned} & - \int \langle H, W \rangle \Delta \eta d\mu \\ & \leq - \int \langle H, \Delta H \rangle \langle D\tilde{\eta} \circ f, H \rangle d\mu + \frac{c}{\varrho^2} \int |H| |\Delta H| \gamma^2 d\mu \\ & \quad + \frac{c}{\varrho} \int |A^\circ|^2 |A|^3 \gamma^3 d\mu + \frac{c}{\varrho^2} \int |A^\circ|^2 |A|^2 \gamma^2 d\mu. \end{aligned}$$

The first integral is estimated by (45), the third integral by (44). Furthermore

$$\begin{aligned} \frac{c}{\varrho^2} \int |H| |\Delta H| \gamma^2 d\mu &\leq \varepsilon \int |\nabla^2 A|^2 \gamma^4 d\mu + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A|^2 d\mu \\ \frac{c}{\varrho^2} \int |A^\circ|^2 |A|^2 \gamma^2 d\mu &\leq \varepsilon \int |A^\circ|^4 |A|^2 \gamma^4 d\mu + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A|^2 d\mu. \end{aligned}$$

The third integral on the right of (35) satisfies

$$\int |\nabla \eta| |\nabla H| |W| d\mu \leq \varepsilon \int |W|^2 \gamma^4 d\mu + \frac{c(\varepsilon)}{\varrho^2} \int |\nabla H|^2 \gamma^2 d\mu$$

and the right hand side is already estimated. Thus putting things together we have shown

$$\begin{aligned} \partial_t \left( \int \frac{1}{2} |H|^2 \eta d\mu \right) + \frac{3}{4} \int |W|^2 \eta d\mu \\ \leq \varepsilon \int (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) \eta d\mu + \frac{c(\varepsilon)}{\varrho^4} \int_{[\gamma>0]} |A|^2 d\mu. \end{aligned}$$

Now (43) follows from Proposition 2.6.

q.e.d.

**Proposition 3.4.** *Let  $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^n$  be a Willmore flow with  $\int_\Sigma |A|^2 d\mu \leq \varkappa$ . There exist constants  $\varepsilon_1 = \varepsilon_1(n) > 0$  and  $c_1 = c(n)/\varkappa > 0$ , such that if  $\varrho > 0$  is chosen with*

$$(46) \quad \int_{\Sigma_\varrho} |A^\circ|^2 d\mu \leq \varepsilon < \varepsilon_1 \quad \text{at time } t = 0 \quad \text{for all } \Sigma_\varrho = \Sigma_\varrho(x_0) \subset \mathbb{R}^n,$$

then for any time  $0 \leq t < t_1 = \min\{c_1 \varrho^4, T\}$  we have

$$(47) \quad \int_{\Sigma_\varrho} |A^\circ|^2 d\mu + \int_0^t \int_{\Sigma_\varrho} (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) d\mu d\tau \\ \leq c(\varepsilon + \varkappa \varrho^{-4} t),$$

$$(48) \quad \int_0^t \|A^\circ\|_{L^\infty(\Sigma_\varrho)}^4 d\tau \leq c(\varepsilon + \varkappa \varrho^{-4} t).$$

Moreover, for  $0 < \sigma \leq \varrho$  and  $\tau < \min\{c_1 \sigma^4, T\}$  we then also have

$$(49) \quad \int_{\Sigma_{\sigma/2}(x)} |A|^2 d\mu \Big|_{t=\tau} \leq \int_{\Sigma_{\sigma}(x)} |A|^2 d\mu \Big|_{t=0} + c \varkappa \sigma^{-4} \tau \quad \forall x \in \mathbb{R}^n.$$

*Proof.* Let  $N = N(n)$  be the number of balls  $B_{\varrho/2} \subset \mathbb{R}^n$  needed to cover  $B_{\varrho} \subset \mathbb{R}^n$  and choose  $\varepsilon_1 \leq \frac{\varepsilon_0}{4N}$ , where  $\varepsilon_0 > 0$  is as in Lemma 3.3. Assume (41) is satisfied on  $[0, t]$  for all  $B_{\varrho} \subset \mathbb{R}^n$ , and integrate (42) to obtain using (46)

$$\begin{aligned} \int_{\Sigma_{\varrho/2}} |A^\circ|^2 d\mu + c_0 \int_0^t \int_{\Sigma_{\varrho/2}} (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) d\mu ds \\ \leq \varepsilon + c \varkappa \varrho^{-4} t. \end{aligned}$$

Assuming  $t \leq c_1 \varrho^4$  where  $c_1$  is chosen with  $0 < c_1 \leq \frac{\varepsilon_0}{4N c \varkappa}$ , we conclude

$$\begin{aligned} \int_{\Sigma_{\varrho}} |A^\circ|^2 d\mu + c_0 \int_0^t \int_{\Sigma_{\varrho}} (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^\circ|^2) d\mu ds \\ \leq N(\varepsilon + c \varkappa \varrho^{-4} t) \\ \leq N(\varepsilon_1 + c \varkappa c_1) \\ \leq \frac{\varepsilon_0}{2}. \end{aligned}$$

It follows that (41) holds up to time  $t = t_1$  for all  $x_0 \in \mathbb{R}^n$ , and (47) follows. In particular  $\int_{\Sigma_{\varrho}} |A^\circ|^2 d\mu \leq c(\varepsilon_1 + \varkappa c_1)$ , whence a covering argument with possibly smaller  $\varepsilon_1, c_1$  implies the smallness hypothesis in Theorem 2.9 for any ball  $B_{2\varrho} \subset \mathbb{R}^n$  and any  $t \in [0, t_1]$ . Inequality (48) now follows from combining (25) with (47), again using a covering. Finally (49) is obtained by integrating (43) and (42) on  $[0, t]$ . q.e.d.

We next state a version of the higher order estimates obtained in [5] which is localized in time.

**Theorem 3.5** (Interior Estimates). *Let  $f : \Sigma \times (0, T] \rightarrow \mathbb{R}^n$  be a Willmore flow satisfying the condition*

$$(50) \quad \sup_{0 < t \leq T} \int_{\Sigma_{\varrho}(0)} |A|^2 d\mu \leq \varepsilon < \varepsilon_0(n),$$

where  $T \leq c(n) \varrho^4$ . Then for any  $k \in \mathbb{N}_0$  we have at time  $t \in (0, T]$  the estimates

$$(51) \quad \|\nabla^k A\|_{L^\infty(\Sigma_{\varrho/2}(0))} \leq c(k) \sqrt{\varepsilon} t^{-\frac{k+1}{4}}$$

$$(52) \quad \|\nabla^k A\|_{L^2(\Sigma_{\varrho/2}(0))} \leq c(k) \sqrt{\varepsilon} t^{-\frac{k}{4}}.$$

*Proof.* By scaling  $f_\varrho(p, t) = \frac{1}{\varrho} f(p, \varrho^4 t)$  we may assume  $\varrho = 1$ . Using (4.13) and (4.9) from [5], see also Proposition 4.6 in [5], we obtain on  $B = B_{3/4}(0)$  the inequalities

$$(53) \quad \int_0^T \int_{\Sigma_{3/4}} (|\nabla^2 A|^2 + |A|^6) d\mu dt \leq c\varepsilon,$$

$$(54) \quad \int_0^T \|A\|_{L^\infty(\Sigma_{3/4})}^4 dt \leq c\varepsilon.$$

Fix a cutoff function  $\tilde{\gamma} \in C_c^\infty(\mathbb{R}^n)$  with  $\chi_{B_{1/2}} \leq \tilde{\gamma} \leq \chi_B$  and  $\|D\tilde{\gamma}\|_{L^\infty} + \|D^2\tilde{\gamma}\|_{L^\infty} \leq c$ . Also define cutoff functions in time by

$$\chi_j(t) = \begin{cases} 0 & \text{for } t \leq (j-1) \frac{T}{m} \\ \frac{m}{T} (t - (j-1) \frac{T}{m}) & \text{in between} \\ 1 & \text{for } t \geq j \frac{T}{m}, \end{cases}$$

where  $0 \leq j \leq m$  and  $m \in \mathbb{N}_0$ . Note  $\chi_0 \equiv 1$  on  $[0, T]$ ,  $\chi_m(T) = 1$  and

$$(55) \quad 0 \leq \dot{\chi}_j \leq \frac{m}{T} \chi_{j-1}.$$

Introducing the notation  $\alpha(t) = \|A\|_{L^\infty(\Sigma_{3/4})}^4$ ,  $E_j(t) = \int |\nabla^{2j} A|^2 \gamma^{4j+4} d\mu$  (where  $\gamma = \tilde{\gamma} \circ f$ ), we have by (4.14) in [5]

$$\frac{d}{dt} E_j(t) + \frac{1}{2} E_{j+1}(t) \leq c\alpha(t) E_j(t) + c(1 + \alpha(t))\varepsilon.$$

Letting  $e_j(t) = \chi_j(t) E_j(t)$  this implies, using also (55),

$$(56) \quad \begin{aligned} \frac{d}{dt} e_j(t) &\leq c\alpha(t) e_j(t) - \frac{1}{2} \chi_j(t) E_{j+1}(t) \\ &\quad + c(1 + \alpha(t))\varepsilon + \frac{m}{T} \chi_{j-1}(t) E_j(t). \end{aligned}$$

We now prove by induction for  $0 \leq j \leq m$  and all  $t \in (0, T]$  the inequality

$$e_j(t) + \frac{1}{2} \int_0^t \chi_j(s) E_{j+1}(s) ds \leq \frac{c(m)\varepsilon}{T^j}.$$

For  $j = 0$  this follows from assumption (50) and estimate (53). Integrating (56) on  $[0, T]$  yields, for  $j \geq 1$ ,

$$\begin{aligned} e_j(t) + \frac{1}{2} \int_0^t \chi_j(s) E_{j+1}(s) ds \\ \leq c \int_0^t \alpha(s) e_j(s) ds + c\varepsilon \int_0^t (1 + \alpha(s)) ds \\ + \frac{m}{T} \int_0^t \chi_{j-1}(s) E_j(s) ds. \end{aligned}$$

Now since  $\int_0^T \alpha(t) dt \leq c\varepsilon$  by (54), we may apply Gronwall's inequality to get

$$\begin{aligned} e_j(t) + \frac{1}{2} \int_0^t \chi_j(s) E_{j+1}(s) ds &\leq c\varepsilon + \frac{cm}{T} \frac{c(m)\varepsilon}{T^{j-1}} \\ &\leq \frac{c(m)\varepsilon}{T^j}, \end{aligned}$$

as  $T \leq c(n)$  by assumption. Thus we have at time  $t = T$

$$\int |\nabla^{2m} A|^2 \gamma^{4m+4} d\mu \leq \frac{c(m)\varepsilon}{T^m}.$$

The estimate for  $\nabla^{2m+1} A$  follows by interpolation as in Lemma 5.1 of [5], taking  $r = 1, p = q = 2, \alpha = 1, \beta = 0, s = 4m + 6$  and  $t = \frac{1}{s} \in [-\frac{1}{2}, \frac{1}{2}]$  there and using again  $T \leq c(n)$ . Renaming  $T$  into  $t$ , the  $L^2$ -estimate (52) is proved. Using (4.9) and (4.7) in [5], the  $L^\infty$ -estimate (51) follows. q.e.d.

#### 4. Construction of the blowup

In this section we rescale the Willmore flow at an assumed singularity at finite or infinite time, thereby constructing a static Willmore surface as a limit. We shall need the following local area bound due to L. Simon [8].

**Lemma 4.1.** *Let  $f : \Sigma \rightarrow \mathbb{R}^n$  be a properly immersed surface. Then for  $0 < \sigma \leq \varrho < \infty$  and  $\Sigma_\varrho = \Sigma_\varrho(x_0)$  one has*

$$\frac{\mu(\Sigma_\sigma)}{\sigma^2} \leq c \left( \frac{\mu(\Sigma_\varrho)}{\varrho^2} + \int_{\Sigma_\varrho} |H|^2 d\mu \right).$$

In particular if  $\Sigma$  is compact without boundary

$$\frac{\mu(\Sigma_\sigma)}{\sigma^2} \leq c (\mathcal{W}(f) + 4\pi\chi(\Sigma)).$$

The following compactness theorem, whose proof is omitted, is a localized version of the result of J. Langer [6].

**Theorem 4.2.** *Let  $f_j : \Sigma_j \rightarrow \mathbb{R}^n$  be a sequence of proper immersions, where  $\Sigma_j$  is a two-dimensional manifold without boundary. Let*

$$\Sigma_j(R) = \{p \in \Sigma_j : |f_j(p)| < R\}$$

and assume the bounds

$$\begin{aligned} \mu_j(\Sigma_j(R)) &\leq c(R) \quad \text{for any } R > 0, \\ \|\nabla^k A_j\|_{L^\infty} &\leq c(k) \quad \text{for any } k \in \mathbb{N}_0. \end{aligned}$$

Then there exists a proper immersion  $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$ , where  $\hat{\Sigma}$  is again a two-manifold without boundary, such that after passing to a subsequence we have a representation

$$(57) \quad f_j \circ \varphi_j = \hat{f} + u_j \quad \text{on } \hat{\Sigma}(j) = \{p \in \hat{\Sigma} : |\hat{f}(p)| < j\}$$

with the following properties:

$$\begin{aligned} \varphi_j : \hat{\Sigma}(j) &\rightarrow U_j \subset \Sigma_j \quad \text{is diffeomorphic,} \\ \Sigma_j(R) &\subset U_j \quad \text{if } j \geq j(R), \\ u_j &\in C^\infty(\hat{\Sigma}(j), \mathbb{R}^n) \quad \text{is normal along } \hat{f}, \\ \|\hat{\nabla}^k u_j\|_{L^\infty(\hat{\Sigma}(j))} &\rightarrow 0 \quad \text{as } j \rightarrow \infty, \text{ for any } k \in \mathbb{N}_0. \end{aligned}$$

Roughly speaking, the theorem says that on any ball  $B_R(0)$  the immersion  $f_j$  can be written as a normal graph  $\hat{f} + u_j$  with small norm for  $j$  large over a limit immersion  $\hat{f}$ , after suitably reparametrizing with  $\varphi_j$ .

Now let  $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^n$  be a smooth Willmore flow defined on a closed surface  $\Sigma$ , where  $0 < T \leq \infty$ . Define

$$\varkappa(r, t) = \sup_{x \in \mathbb{R}^n} \int_{\Sigma_r(x)} |A(t)|^2 d\mu_t.$$

Choose an arbitrary sequence  $r_j \searrow 0$  and assume concentration in the sense that for all  $j$

$$(58) \quad t_j = \inf\{t \geq 0 : \varkappa(r_j, t) > \varepsilon_1\} < T,$$



where  $\varepsilon_1 = \varepsilon_0/c$  and  $\varepsilon_0, c$  are the constants from Theorem 1.2 in [5]. Clearly

$$\int_{\Sigma_{r_j}(x)} |A(t_j)|^2 d\mu_{t_j} \leq \varepsilon_1 \quad \text{for any } x \in \mathbb{R}^n.$$

On the other hand, choosing an appropriate sequence of balls at times  $\tau_\nu \searrow t_j$ , we find a point  $x_j \in \mathbb{R}^n$  satisfying

$$\int_{f^{-1}(\overline{B_{r_j}(x_j)})} |A(t_j)|^2 d\mu_{t_j} \geq \varepsilon_1.$$

Now we rescale by considering

$$(59) \quad \begin{aligned} f_j &: \Sigma \times [-r_j^{-4}t_j, r_j^{-4}(T - t_j)] \rightarrow \mathbb{R}^n, \\ f_j(p, t) &= \frac{1}{r_j}(f(p, t_j + r_j^4 t) - x_j). \end{aligned}$$

By the above we have  $\varkappa_j(1, t) \leq \varepsilon_1$  for  $t \leq 0$  and

$$(60) \quad \int_{f^{-1}(\overline{B_1(0)})} |A_j(0)|^2 d\mu_j \geq \varepsilon_1.$$

Furthermore Theorem 1.2 of [5] yields  $r_j^{-4}(T - t_j) \geq c_0$  and in fact

$$\varkappa_j(1, t) \leq \varepsilon_0 \quad \text{for } 0 < t \leq c_0.$$

We may now apply Theorem 3.5 on parabolic cylinders  $B_1(x) \times (t - 1, t]$  to obtain

$$(61) \quad \|\nabla^k A_j(t)\|_{L^\infty} \leq c(k) \quad \text{for } -r_j^{-4}t_j + 1 \leq t \leq c_0.$$

Furthermore Lemma 4.1 yields

$$\frac{\mu_j(t)(\Sigma_R(0))}{R^2} \leq c(\mathcal{W}(f_0) + 4\pi \chi(\Sigma)) < \infty.$$

We apply Theorem 4.2 to the sequence  $f_j = f_j(\cdot, 0) : \Sigma \rightarrow \mathbb{R}^n$ , thus obtaining a limit immersion  $\hat{f}_0 : \hat{\Sigma} \rightarrow \mathbb{R}^n$ . Let  $\varphi_j : \hat{\Sigma}(j) \rightarrow U_j \subset \Sigma$  be as in (57). Then the reparametrization

$$(62) \quad f_j(\varphi_j, \cdot) : \hat{\Sigma}(j) \times [0, c_0] \rightarrow \mathbb{R}^n$$

is a Willmore flow with initial data

$$(63) \quad f_j(\varphi_j, 0) = \hat{f}_0 + u_j : \hat{\Sigma}(j) \rightarrow \mathbb{R}^n.$$

The flows (62) satisfy the curvature bounds (61) and have initial data converging locally in  $C^k$  to the immersion  $\hat{f}_0 : \Sigma \rightarrow \mathbb{R}^n$ . By standard estimates for geometric evolution equations, see (4.24)–(4.28) in [5], we deduce the locally smooth convergence

$$(64) \quad f_j(\varphi_j, \cdot) \rightarrow \hat{f},$$

where  $\hat{f} : \hat{\Sigma} \times [0, c_0] \rightarrow \mathbb{R}^n$  is a Willmore flow with initial data  $f_0$ . But on the other hand we have

$$\begin{aligned} & \int_0^{c_0} \int_{\hat{\Sigma}(j)} |\mathbf{W}(f_j(\varphi_j, t))|^2 d\mu_{f_j(\varphi_j, \cdot)} dt \\ &= \int_0^{c_0} \int_{U_j} |\mathbf{W}(f_j(\cdot, t))|^2 d\mu_j dt \\ &\leq \int_{\Sigma} |A_j(c_0)|^2 d\mu_j - \int_{\Sigma} |A_j(0)|^2 d\mu \\ &= \int_{\Sigma} |A(t_j + r_j^4 c_0)|^2 d\mu - \int_{\Sigma} |A(t_j)|^2 d\mu, \end{aligned}$$

which converges to zero as  $j \rightarrow \infty$ . This implies that  $\mathbf{W}(\hat{f}) \equiv 0$  which means that  $\hat{f}(\cdot, t) \equiv \hat{f}_0$  is an immersed Willmore surface, which is independent of time. Furthermore (60) implies, because of the smooth convergence in (64),

$$(65) \quad \int_{\hat{f}^{-1}(\overline{B_1(0)})} |\hat{A}|^2 d\hat{\mu} \geq \varepsilon_1 > 0.$$

Thus  $\hat{f}$  is not a union of planes.

**Lemma 4.3.** *Let  $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$  be the blowup constructed above. If  $\hat{\Sigma}$  contains a compact component  $C$ , then in fact  $\hat{\Sigma} = C$  and  $\Sigma$  is diffeomorphic to  $C$ .*

*Proof.* For  $j$  sufficiently large,  $\varphi_j(C)$  is open and closed in  $\Sigma$ . Hence by connectedness of  $\Sigma$  we have  $\Sigma = \varphi_j(C)$  and thus  $\hat{\Sigma} = C$ . q.e.d.

**Theorem 4.4** (Nontriviality of the Blowup). *Let  $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$  be the blowup of a Willmore flow as constructed above. Then none of*

the components of  $\hat{f}$  parametrizes a round sphere. In particular, the blowup has a component which is a nonumbilic (compact or noncompact) Willmore surface.

*Proof.* Otherwise, Lemma 4.3 implies that the blowup surface  $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$  is an embedded round sphere, i.e., has no further components. It follows that, up to the diffeomorphism  $\varphi_j : \hat{\Sigma} \rightarrow \Sigma$ , the map  $f_j(\cdot, 0)$  is  $C^k$ -close to a round sphere and therefore

$$\begin{aligned} \int_{\Sigma} |A^\circ(t_j)|^2 d\mu &= \int_{\Sigma} |A_j^\circ(0)|^2 d\mu_j \rightarrow 0, \\ \mu(t_j)(\Sigma) &= r_j^2 \mu_j(0)(\Sigma) \rightarrow 0. \end{aligned}$$

This contradicts the lower area bound which will be proved in Theorem 5.2. q.e.d.

### 5. Small initial energy

In this section we finally prove our main result:

**Theorem 5.1** (Global Existence and Convergence for Small Initial Energy). *There exists an  $\varepsilon_0 = \varepsilon_0(n) > 0$  such that if at time  $t = 0$  there holds*

$$\mathcal{W}(f_0) = \int_{\Sigma} |A^\circ|^2 d\mu < \varepsilon_0,$$

*then the Willmore flow exists smoothly for all times and converges exponentially to a round sphere as  $t \rightarrow \infty$ .*

We split the proof into several steps. The first step was already used in Theorem 4.4 and is of independent interest.

**Theorem 5.2** (Area Estimate). *Let  $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^n$  be a Willmore flow with  $\mathbf{W}(f) = \int_{\Sigma} |A^\circ|^2 d\mu \leq \varepsilon < \varepsilon_1$ , where  $\varepsilon_1 = \varepsilon_1(n)$  is as in Proposition 3.4. Then*

$$(66) \quad (1 - c\varepsilon) \mu_0(\Sigma) \leq \mu(\Sigma) \leq (1 + c\varepsilon) \mu_0(\Sigma)$$

$$(67) \quad \int_0^t \int_{\Sigma} (|\nabla A|^2 + |A|^2 |A^\circ|^2) d\mu ds \leq c\varepsilon \mu_0(\Sigma).$$

*Proof.* We have

$$\frac{d}{dt} \mu(\Sigma) = - \int_{\Sigma} |\nabla H|^2 d\mu + \int_{\Sigma} \langle Q(A^\circ)H, H \rangle d\mu.$$

Multiplying Simons' identity (10) by  $A^\circ$  and integrating yields (cf. Lemma 2.2):

$$(68) \quad \begin{aligned} 2 \int_{\Sigma} |\nabla A^\circ|^2 d\mu + \int_{\Sigma} |H|^2 |A^\circ|^2 d\mu \\ = \int_{\Sigma} |\nabla H|^2 d\mu + \int_{\Sigma} A^\circ * A^\circ * A^\circ * A^\circ d\mu. \end{aligned}$$

As  $\langle Q(A^\circ)H, H \rangle \leq |A^\circ|^2 |H|^2$  by (14), we obtain

$$\begin{aligned} \frac{d}{dt} \mu(\Sigma) + 2 \int_{\Sigma} |\nabla A^\circ|^2 d\mu &\leq c \int_{\Sigma} |A^\circ|^4 d\mu \\ &\leq c \|A^\circ\|_{L^\infty}^4 \mu(\Sigma). \end{aligned}$$

From (48) with  $\varrho = \infty$  we have

$$\int_0^t \|A^\circ\|_{L^\infty}^4 ds \leq c\varepsilon,$$

and the Gronwall inequality yields

$$\mu(\Sigma) + 2 \int_0^t \int_{\Sigma} |\nabla A^\circ|^2 d\mu ds \leq (1 + c\varepsilon) \mu_0(\Sigma).$$

On the other hand, by Michael-Simon

$$\begin{aligned} \int |\nabla H|^2 d\mu &\leq c \left( \int (|\nabla^2 H| + |H| |\nabla H|) d\mu \right)^2 \\ &\leq c \mu(\Sigma) \int_{\Sigma} (|\nabla^2 H|^2 + |H|^2 |\nabla H|^2) d\mu. \end{aligned}$$

As  $\langle Q(A^\circ)H, H \rangle \geq 0$  we obtain

$$\frac{d}{dt} \mu(\Sigma) \geq -c \mu(\Sigma) \int_{\Sigma} (|\nabla^2 H|^2 + |H|^2 |\nabla H|^2) d\mu.$$

Using (47) with  $\varrho = \infty$  implies the remaining inequality in (66). In particular we obtain

$$\int_0^t \int_{\Sigma} |\nabla A^\circ|^2 d\mu ds \leq c\varepsilon \mu_0(\Sigma),$$

and

$$\begin{aligned} \int_0^t \int_{\Sigma} |A^\circ|^4 d\mu ds &\leq (1 + c\varepsilon) \mu_0(\Sigma) \int_0^t \|A^\circ\|_{L^\infty}^4 ds \\ &\leq c\varepsilon \mu_0(\Sigma). \end{aligned}$$

Finally from (68) and Codazzi (5)

$$\begin{aligned} \int_0^t \int_{\Sigma} |H|^2 |A^\circ|^2 d\mu ds &\leq c \int_0^t \int_{\Sigma} (|\nabla A^\circ|^2 + |A^\circ|^4) d\mu ds \\ &\leq c\varepsilon \mu_0(\Sigma). \end{aligned}$$

This proves the theorem.

q.e.d.

**Remark.** The extrinsic diameter is bounded above and below by the diameter of the initial surface, cf. [8].

**Lemma 5.3.** *Under the assumptions of Theorem 5.1 there exists a radius  $r_0 > 0$  such that*

$$\int_{\Sigma_{r_0}(x)} |A(t)|^2 d\mu \leq \varepsilon_1 \quad \text{for all } x \in \mathbb{R}^n, t \in [0, \infty),$$

where  $\varepsilon_1 > 0$  is as in (58).

*Proof.* Otherwise, the blowup construction of Section 4 yields an immersed Willmore surface  $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$  with

$$\int_{\hat{f}^{-1}(\overline{B_1(0)})} |\hat{A}|^2 d\hat{\mu} \geq \varepsilon_1,$$

whereas

$$\int_{\hat{\Sigma}} |\hat{A}^\circ|^2 d\hat{\mu} < \varepsilon_0.$$

By Theorem 2.7 the surface  $\hat{f}$  must be a union of embedded planes and round spheres, which however contradicts the nontriviality of the blowup, Theorem 4.4. q.e.d.

**Lemma 5.4.** *For any sequence  $t_j \rightarrow \infty$  there exist  $x_j \in \mathbb{R}^n$  and  $\varphi_j \in \text{Diff}(\Sigma)$  such that, after passing to a subsequence, the immersions  $f(\varphi_j, t_j) - x_j$  converge smoothly to an embedded round sphere.*

*Proof.* Let  $x_j = f(p, t_j)$  where  $p \in \Sigma$  is arbitrary. By the previous lemma and the interior curvature estimates from Theorem 3.5, we have for  $t_j \geq 1$

$$(69) \quad \|\nabla^k A(\cdot, t_j)\|_{L^\infty} \leq c(k).$$

Furthermore, Lemma 4.1 yields the area bound

$$\frac{\mu(t_j)(B_R(x_j))}{R^2} \leq c(\mathcal{W}(f(\cdot, t_j)) + 4\pi\chi(\Sigma)).$$

According to Theorem 4.2, there exist a properly immersed surface  $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$  and diffeomorphisms  $\varphi_j : \hat{\Sigma}(j) \rightarrow U_j \subset \Sigma$ , such that (after selection of a subsequence)

$$f(\varphi_j, t_j) - x_j \rightarrow \hat{f}$$

locally in  $C^k$  on  $\hat{\Sigma}$ . On  $\hat{\Sigma}(j)$  we consider the Willmore flows

$$g_j(p, t) = f(\varphi_j(p), t_j + t) - x_j \quad (t \geq -t_j).$$

These satisfy the curvature estimates (69), and the initial data (at  $t = 0$ ) converge to  $\hat{f}$ . Arguing as in (64), we obtain that  $g_j$  converges locally smoothly on  $\hat{\Sigma} \times [0, \infty)$  to a Willmore flow  $g : \hat{\Sigma} \times [0, \infty) \rightarrow \mathbb{R}^n$  with initial data  $\hat{f}$ . But now

$$\int_0^1 \int_{\hat{\Sigma}(j)} |\mathbf{W}(g_j)|^2 d\mu_{g_j} dt \leq \int_{t_j}^{t_j+1} \int_{\Sigma} |\mathbf{W}(f)|^2 d\mu dt \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Therefore we have  $\mathbf{W}(g) \equiv 0$  which proves that  $\hat{f}$  is a Willmore surface, and Theorem 2.7 implies that  $\hat{f}$  is a union of embedded planes and round spheres. Using the upper area bound in (66) and excluding several components as in Lemma 4.3 we conclude that  $\hat{f}$  must be a round sphere, and that the subconvergence is smooth. q.e.d.

As a consequence of the above, we obtain that

$$(70) \quad \mathcal{W}(f) = \int_{\Sigma} |A^\circ|^2 d\mu \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Moreover, Theorem 5.2 implies the existence of the nonzero limit

$$(71) \quad \omega = \lim_{t \rightarrow \infty} \mu(t)(\Sigma) \in (0, \infty).$$

Finally we now prove exponential decay of the curvature; from this one obtains smooth convergence of  $f$  to a round sphere  $\hat{f}$  and thus Theorem 5.1 in a standard way.

**Lemma 5.5.** *As  $t \nearrow \infty$ , the following asymptotic statements hold, where  $\lambda > 0$  is a constant:*

$$(72) \quad \|\nabla^k A(t)\|_{L^\infty} \leq c_k e^{-\lambda t} \quad \text{for } k \geq 1,$$

$$(73) \quad \|A^\circ\|_{L^\infty} \leq c_0 e^{-\lambda t}.$$

*Proof.* For  $\omega$  as in (71), the previous Lemma implies that the sectional curvature and the mean curvature of  $f(\cdot, t)$  satisfy

$$\begin{aligned} \left\| K(\cdot, t) - \frac{4\pi}{\omega} \right\|_{L^\infty} &\longrightarrow 0, \\ \left\| |H|^2 - \frac{16\pi}{\omega} \right\|_{L^\infty} &\longrightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

In particular, we may assume after a fixed time translation that

$$|H|^2 \geq c > 0 \quad \text{for all } t \geq 0.$$

By Lemma 3.3, we then have for all  $t$

$$\partial_t \int_{\Sigma} |A^\circ|^2 d\mu + c \int_{\Sigma} (|\nabla^2 A|^2 + |\nabla A|^2 + |A^\circ|^2) d\mu \leq 0,$$

which implies

$$(74) \quad \int_{\Sigma} |A^\circ|^2 d\mu + \int_t^\infty \int_{\Sigma} (|\nabla^2 A|^2 + |\nabla A|^2) d\mu ds \leq c e^{-2\lambda t},$$

for a constant  $\lambda = \lambda(n) > 0$ . From here we easily derive exponential decay of all derivatives of  $A$ . Namely, letting  $\varrho \rightarrow \infty$  in Corollary 3.4 of [5], we have for  $\phi = \nabla^m A$  ( $m \geq 1$ )

$$\frac{d}{dt} \int |\phi|^2 d\mu + \frac{3}{4} \int |\nabla^2 \phi|^2 d\mu \leq \int (P_3^{m+2}(A) + P_5^m(A)) * \phi d\mu.$$

Using that  $A$  and all its derivatives remain bounded as  $t \rightarrow \infty$ , we can estimate

$$\begin{aligned} \int P_2^0(A) * \nabla^{m+2} A * \phi d\mu &\leq \varepsilon \int |\nabla^2 \phi|^2 d\mu + c(\varepsilon) \int |\nabla^m A|^2 d\mu, \\ \int (\hat{P}_3^{m+2}(A) + P_5^m(A)) * \phi d\mu &\leq c \sum_{j=1}^{m+1} \int |\nabla^j A|^2 d\mu. \end{aligned}$$

Here  $\hat{P}_3^{m+2}(A)$  denotes all terms of type  $P_3^{m+2}(A)$  that do not contain the  $(m+2)$ -th derivative, and of course  $c$  is not a universal constant here. We obtain

$$\frac{d}{dt} \int |\phi|^2 d\mu + \frac{1}{2} \int |\nabla^2 \phi|^2 d\mu \leq c \sum_{j=1}^{m+1} \int |\nabla^j A|^2 d\mu,$$

and now by induction using (74)

$$\|\nabla^m A\|_{L^2}^2 + \int_t^\infty \|\nabla^{m+2} A\|_{L^2}^2 ds \leq c e^{-2\lambda t}.$$

By a Sobolev inequality, e.g., the Michael-Simon inequality, we deduce

$$(75) \quad \|A^\circ\|_{L^\infty} \leq c_0 e^{-\lambda t},$$

$$(76) \quad \|\nabla^k A\|_{L^\infty} \leq c_k e^{-\lambda t},$$

which ends the proof of Lemma 5.5 and of Theorem 5.1. q.e.d.

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